Integers

Today

- Numeric Encodings
- Programming Implications
- Basic operations
- Programming Implications

Next time

- Floats

Checkpoint

Encoding integers in binary

- Positive integers, easy

binary to unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \times 2^i$$

- What about negative integers?

Encoding integers in binary

- Idea #1: sign bit
  - use 1 in the most significant (leftmost) bit like a minus sign
    - 3 = 0011, -3 = 1011
  - intuitive, but simple arithmetic is complicated
    - 5 + -3 = 0101 + 1011 = *a miracle occurs* = 0010
- Idea #2: ones' complement
  - flip all bits for negatives
    - 3 = 0011, -3 = 1100
  - addition not too bad (just add and then add carry bit if any)
    - 5 + -3 = 0101 + 1100 = 0001 + 1 (carry) = 0010

Encoding integers

- Both ideas lead to two representations of zero, positive and negative:
  - sign bit: 0000 and 1000
  - ones' complement: 0000 1111
  - 5 + -5 = 0101 + 1010 = 1111 = -0

Encoding integers

- Idea #3: Two's complement
  - Informal encoding view:
  - To encode –N, encode N, flip all bits, add 1
    - 5 = 0101,
    - -5 = 1010 + 1 = 1011
  - More formally, given *w* bits $[x_{w-1}, x_{w-2}, \ldots, x_1, x_0]$,
    - $N = -(2^{w-1}) * x_{w-1} + \sum 2^i * x_i$ for *i* from 0 to *w*-2
    - $1011 = -2^3 + 3 = -8 + 3 = -5$
- Addition is now simple: always add, ignore overflow
  - 5 + -5 = 0101 + 1011 = 0000
- Only one zero (why?)
- Significant bit still serves as sign bit

Encoding integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \times 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \times 2^{w-1} + \sum_{i=0}^{w-2} x_i \times 2^i$$

Sign Bit

C `short` 2 bytes long

```
short int x =  15213;
short int y = -15213;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

Encoding example

```
x = 15213:   00111011 01101101
y = -15213:  11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric ranges

- Unsigned Values
  - Umin = 0
    - 000…0
  - UMax = 2^w-1
    - 111…1
- Two's Complement Values
  - Tmin = -2^{w-1}
    - 100…0
  - TMax = $2^{w-1} - 1$
    - 011…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for other word sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- Observations
  - $|TMin| = |TMax| + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$

C constants

- `#include <limits.h>`
- Declares
  - `ULONG_MAX`
  - `INT_MAX, INT_MIN`
  - `LONG_MAX, LONG_MIN`
- Values platform-specific

Unsigned & signed numeric values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- Equivalence
  - Same encodings for nonnegative values
- Uniqueness (bijections)
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- $\Rightarrow$ Can invert mappings
  - U2B($\boldsymbol{x}$) = B2U^{-1}($\boldsymbol{x}$)
    - Bit pattern for unsigned integer
  - T2B($\boldsymbol{x}$) = B2T^{-1}($\boldsymbol{x}$)
    - Bit pattern for two's comp integer

Casting signed to unsigned

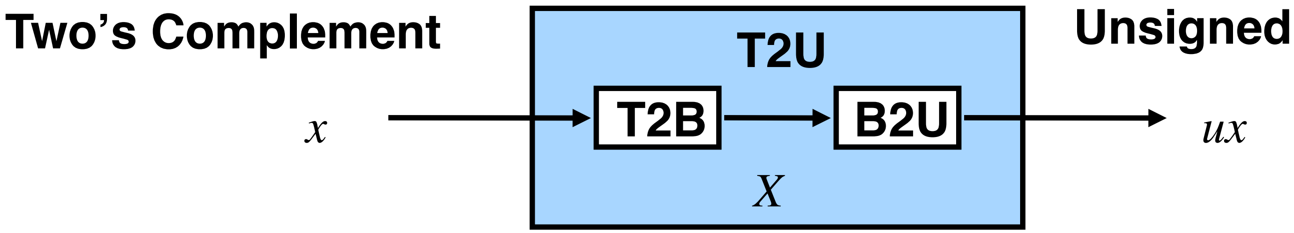
- C allows conversions from signed to unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

- Resulting value
  - No change in bit representation
  - Non-negative values unchanged
    - *ux* = 15213
  - Negative values change into (large) positive values
    - *uy* = 50323

Relation between signed & unsigned

Casting from signed to unsigned



Maintain same bit pattern

Consider B2U and B2T equations

$$B2U(X) = \sum_{i=0}^{w-1} x_i \times 2^i \qquad B2T(X) = -x_{w-1} \times 2^{w-1} + \sum_{i=0}^{w-2} x_i \times 2^i$$

and a bit pattern X; compute $B2U(X) - B2T(X)$
weighted sum of for bits from 0 to w – 2 cancel each other

$$B2U(X) - B2T(X) = x_{w-1}(2^{w-1} - -2^{w-1}) = x_{w-1}2^w$$

$$B2U(X) = x_{w-1}2^w + B2T(X)$$

If we let $B2T(X) = x$

$$B2U(T2B(x)) = T2U(x) = x_{w-1}2^w + x$$

$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation between signed & unsigned

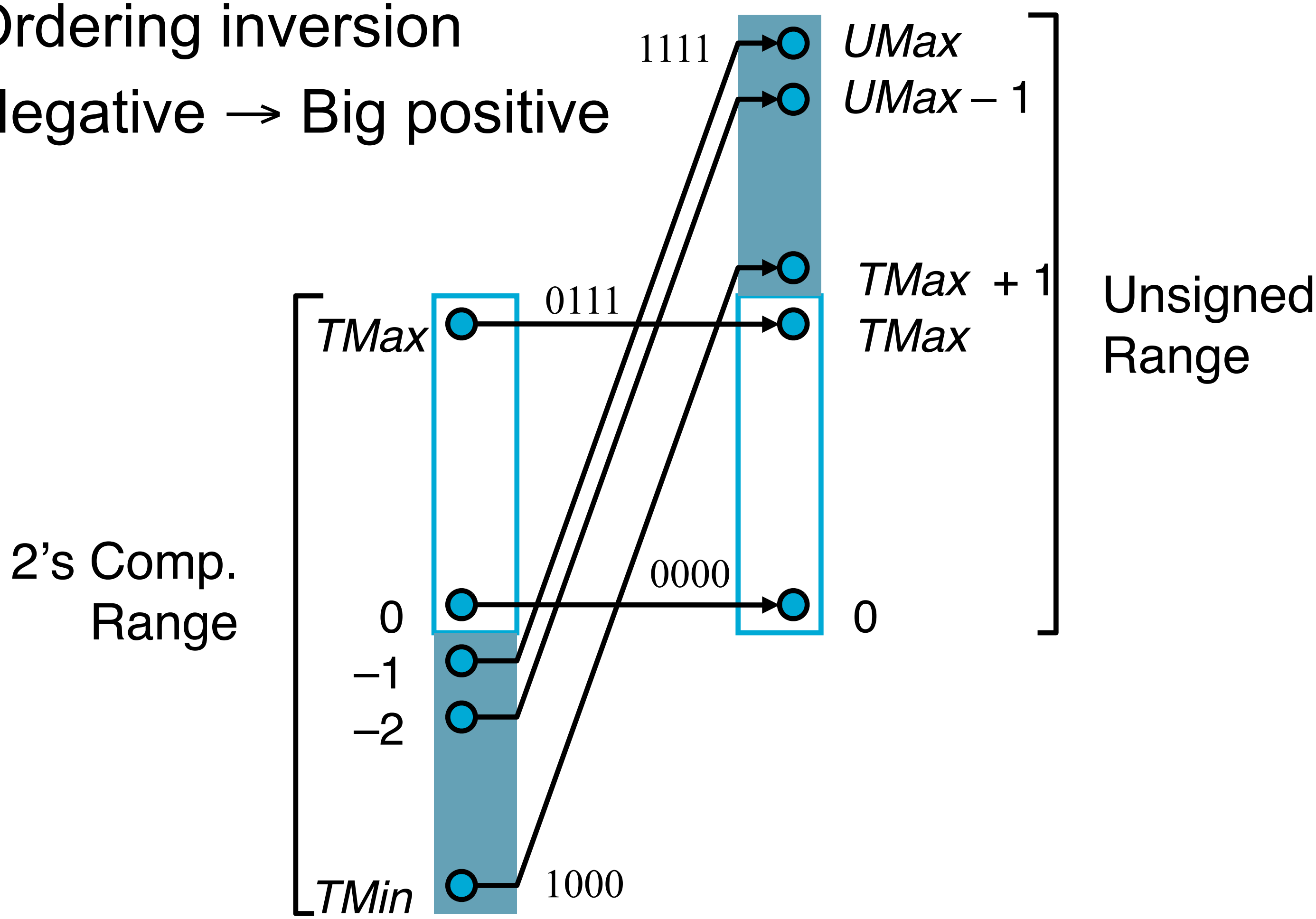
$$T2U(x) = x_{w-1} 2^{w} + x$$

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	*-32768*	1	*32768*
Sum		**-15213**		**50323**

$$\boldsymbol{ux = x + 2^{16} = -15213 + 65536}$$

Conversion - graphically

- 2's Comp. → Unsigned
  - Ordering inversion
  - Negative → Big positive



Signed and unsigned in C

- Constants
  - By default are considered to be signed integers
  - Unsigned if have “U” as suffix

```
0U, 4294967259U
```

- Casting
  - Explicit casting bet/ signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting

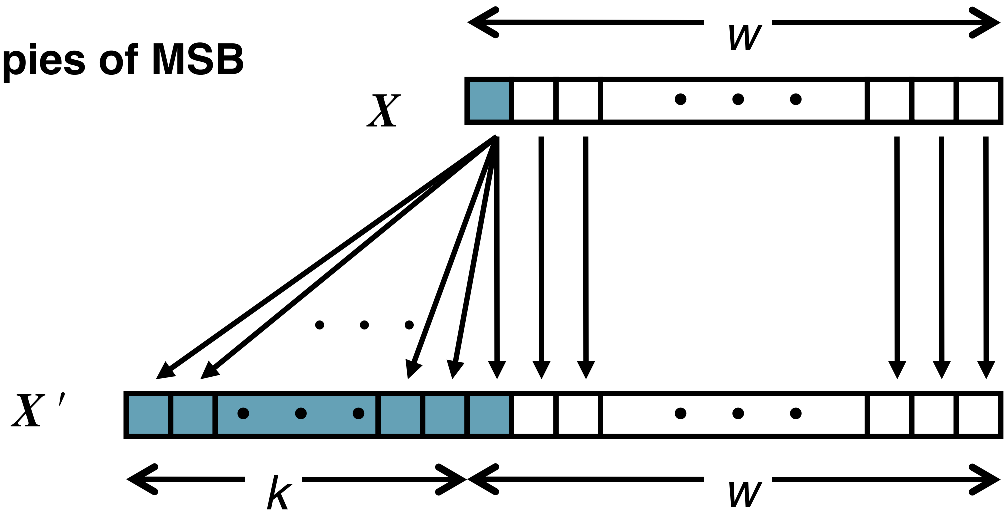
```
tx = ux;
uy = ty;
```

  - Mixed expressions – cast to *unsigned* first

```
tx + ux;
uy < ty;
```

Sign extension

- Task:
  - Given w-bit signed integer x
  - Convert it to w+k-bit integer with same value
- Rule:
  - Make k copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign extension example

- Converting from smaller to larger integer data type
- C automatically performs sign extension

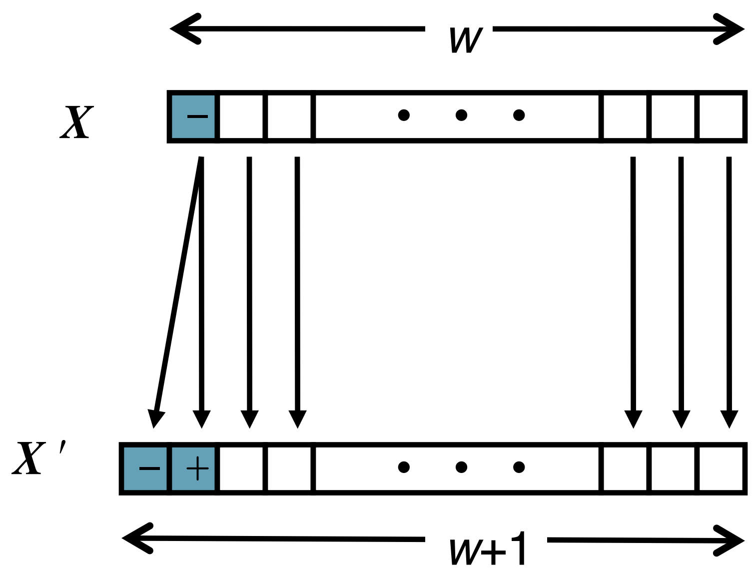
```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

Justification for sign extension

- Prove correctness by induction on k
  - Induction Step: extending by single bit maintains value

$$B2T(X) = -x_{w-1} \times 2^{w-1} + \sum_{i=0}^{w-2} x_i \times 2^i$$



  - Key observation: $-2^w + 2^{w-1} = -2^{w-1}$ =
  - Look at weight of upper bits:
    - X $-2^{w-1}\, x_{w-1}$
    - X' $-2^w\, x_{w-1} + 2^{w-1}\, x_{w-1} = -2^{w-1}\, x_{w-1}$

Why should I use unsigned?

- Don't use just because number nonzero
  - C compilers on some machines generate less efficient code
  - Easy to make mistakes (e.g., casting)
  - Few languages other than C supports unsigned integers
- Do use when need extra bit's worth of range
  - Working right up to limit of word size

Checkpoint

Negating with complement & increment

- Claim: Following holds for 2's complement
  - ~x + 1 == -x
- Complement
  - Observation: ~x + x == 1111…11$_2$ == -1

x	1	0	0	1	1	1	0	1
+ ~x	0	1	1	0	0	0	1	0
-1	1	1	1	1	1	1	1	1

- Increment
  - $\sim x + \cancel{x} + (-\cancel{x} + 1) == -\cancel{1} + (-x + \cancel{1})$
  - ~x + 1 == -x

Comp. & incr. examples

x = 15213

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`~x`	-15214	`C4 92`	`11000100 10010010`
`~x+1`	-15213	`C4 93`	`11000100 1001001`**`1`**
`y`	-15213	`C4 93`	`11000100 10010011`

0

	Decimal	Hex	Binary
`0`	0	`00 00`	`00000000 00000000`
`~0`	-1	`FF FF`	`11111111 11111111`
`~0+1`	0	`00 00`	`00000000 00000000`

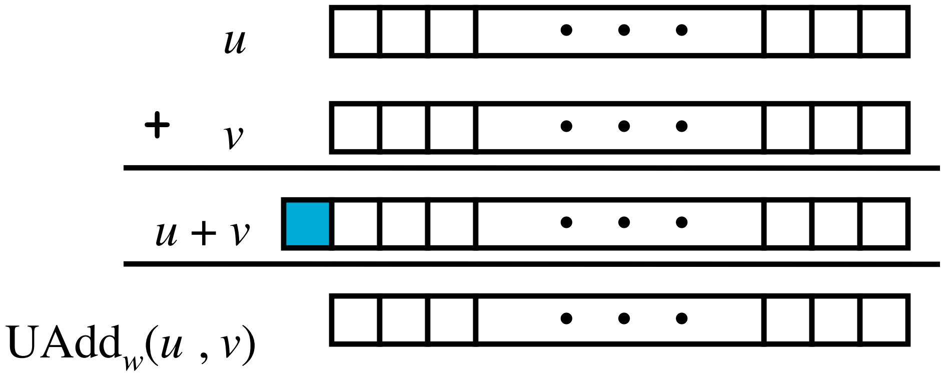
Unsigned addition

- Standard addition function
  - Ignores carry output
- Implements modular arithmetic
  - s = $UAdd_w(u, v) = u + v \mod 2^w$

Operands: *w* bits

True Sum: *w*+1 bits

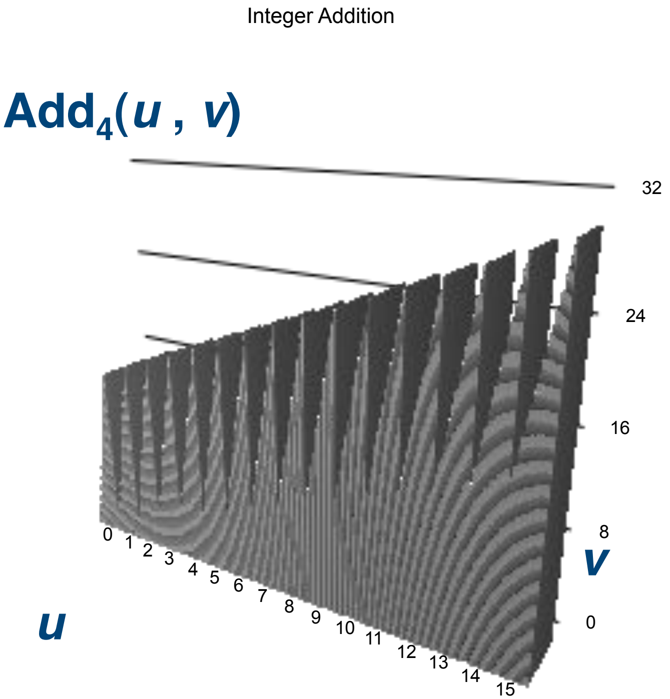
Discard Carry: *w* bits



$$UAdd_w(u,v) = \begin{cases} u + v, u + v < 2^w \\ u + w - 2^w, 2^w \leq x + y < 2^{w+1} \end{cases}$$

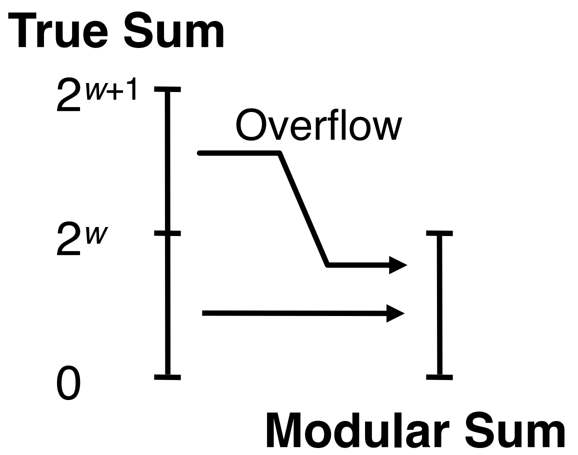
Visualizing integer addition

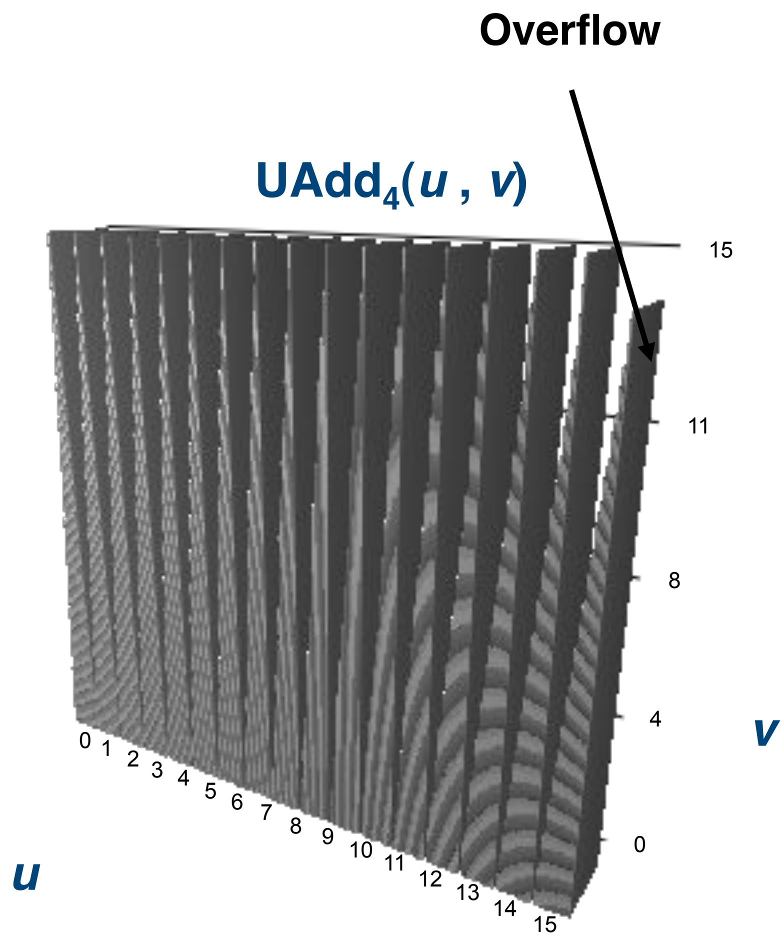
- Integer addition
  - 4-bit integers u, v
  - Compute true sum Add4(u , v)
  - Values increase linearly with u and v
  - Forms planar surface



Visualizing unsigned addition

- Wraps around
  - If true sum ≥ 2^w
  - At most once





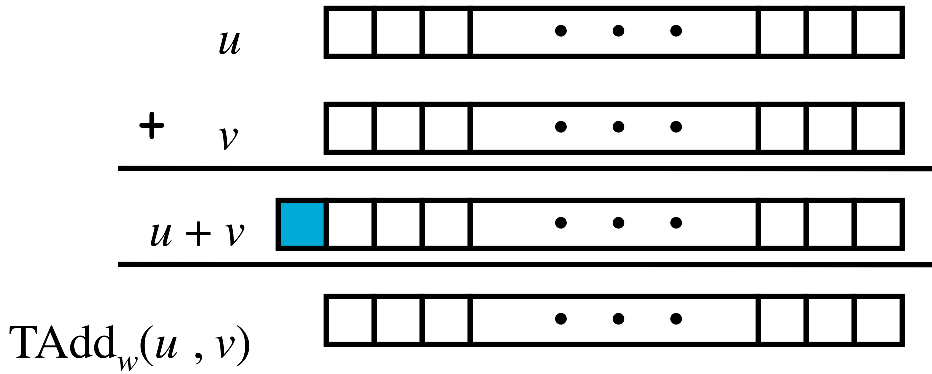
Two’s complement addition

- TAdd and UAdd have identical Bit-level behavior
  - Signed vs. unsigned addition in C:
  - int s, t, u, v;
  - s = (int) ((unsigned) u + (unsigned) v);
  - t = u + v
  - Will give s == t

Operands: *w* bits

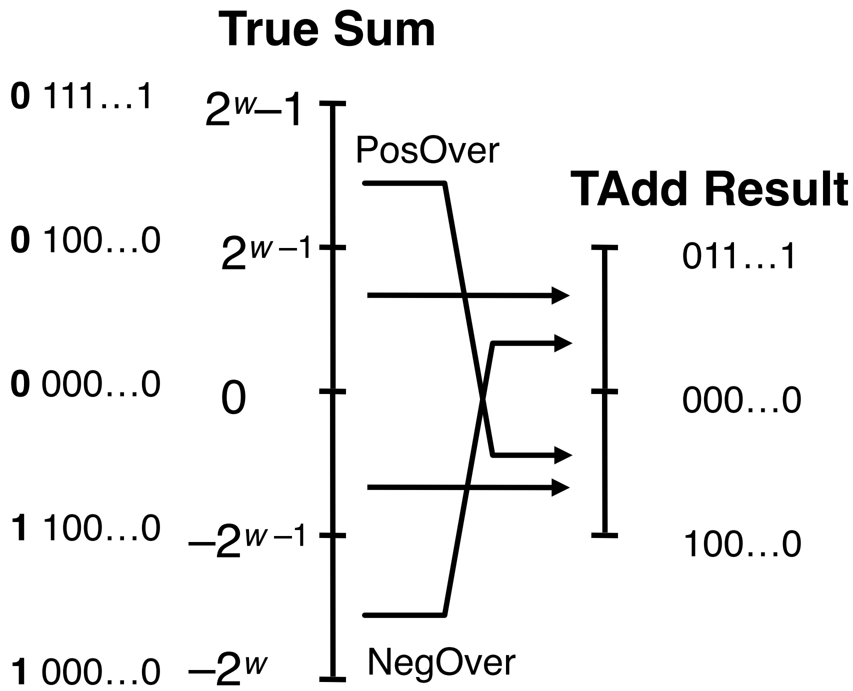
True Sum: *w*+1 bits

Discard Carry: *w* bits



Characterizing TAdd

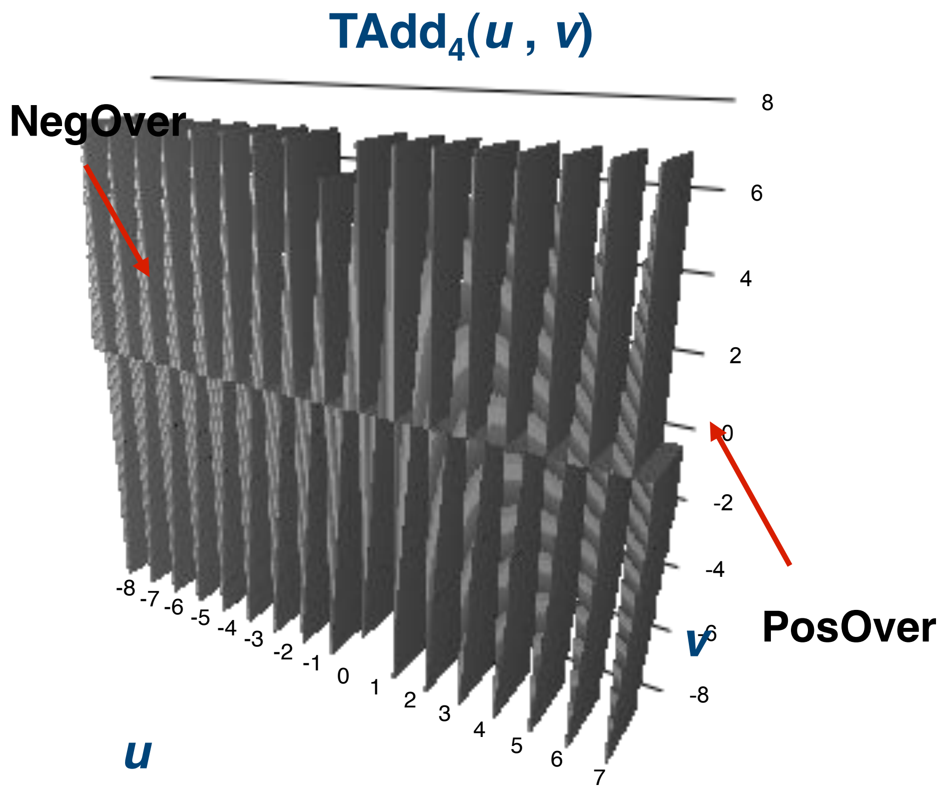
- Functionality
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

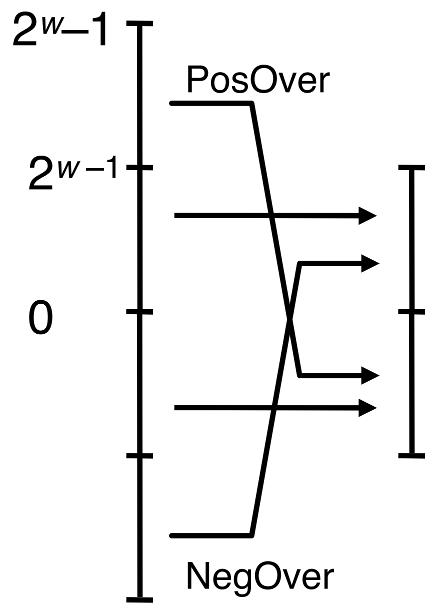
Visualizing 2's comp. addition

- Values
  - 4-bit two's comp.
  - Range from -8 to +7
- Wraps Around
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Detecting 2's comp. overflow

- Task
  - Given s = TAddw(u , v)
  - Determine if s = Addw(u , v)
  - Example
  - int s, u, v;
  - s = u + v;
- Claim
  - Overflow iff either:
    - u, v < 0, s ≥ 0 (NegOver)
    - u, v ≥ 0, s < 0 (PosOver)



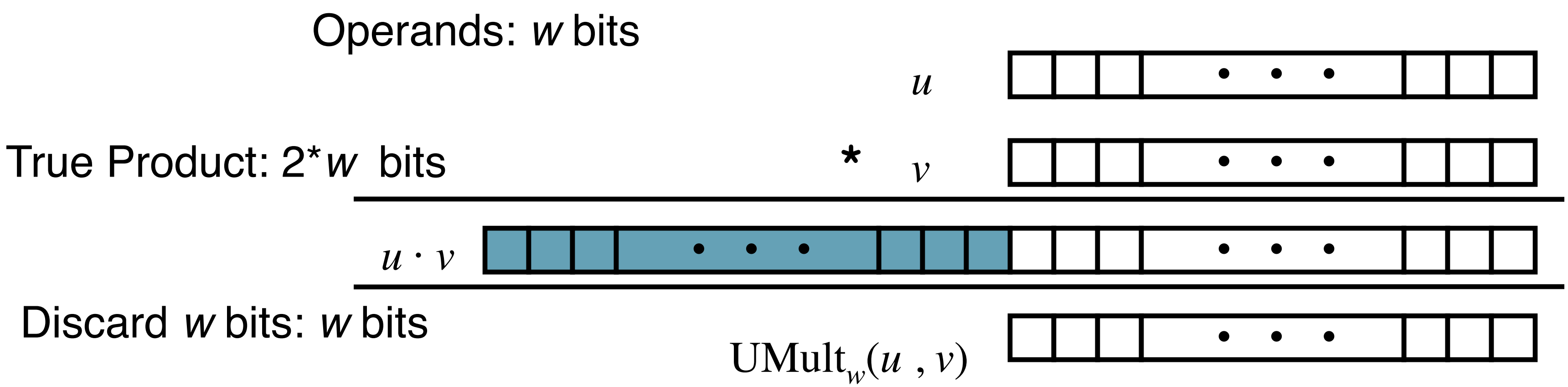

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```

Checkpoint

Multiplication

- Computing exact product of w-bit numbers x, y
  - Either signed or unsigned
- Ranges
  - Unsigned: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
    - May need up to 2w bits to represent
  - Two's complement min: $x * y \geq (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
    - Up to 2^{w-1} bits
  - Two's complement max: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
    - Up to 2w bits
- Maintaining exact results
  - Would need to keep expanding word size with each product computed
  - Done in software by "arbitrary precision" arithmetic packages

Unsigned multiplication in C



- Standard multiplication function
  – Ignores high order ***w*** bits
- Implements modular arithmetic

$$\mathbf{UMult}_w(u\,,\,v) = u \cdot v \mod 2^w$$

Unsigned vs. signed multiplication

- Unsigned multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

  – Truncates product to w-bit number up = $\text{UMult}_w(\text{ux}, \text{uy})$

  – Modular arithmetic: up = ux * uy mod 2^w

- Two's complement multiplication

```
int x, y;
int p = x * y;
```

  – Compute exact product of two w-bit numbers x, y

  – Truncate result to w-bit number p = TMultw(x, y)

Unsigned vs. signed multiplication

- Unsigned multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- Two's complement multiplication

```
int x, y;
int p = x * y;
```

- Relation
  - Signed multiplication gives same bit-level result as unsigned
  - up == (unsigned) p

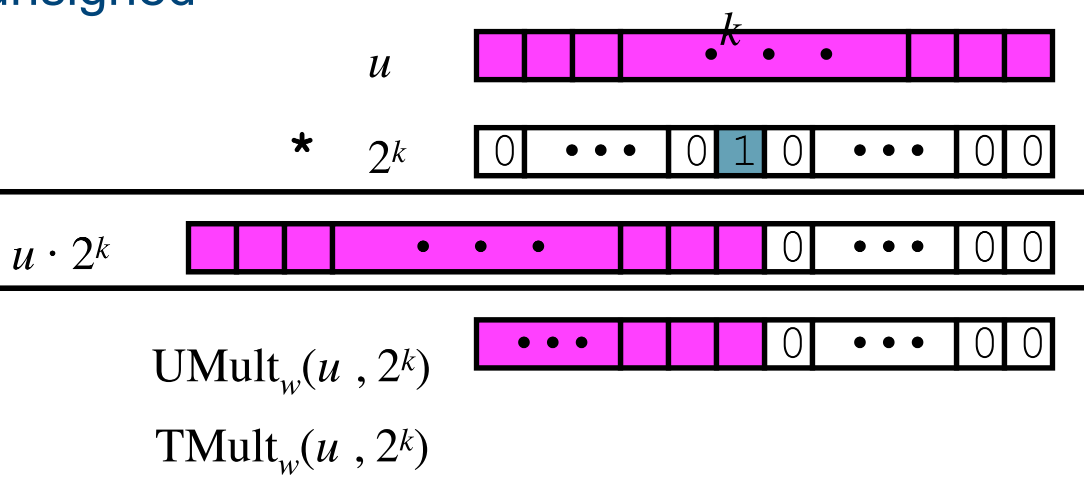
Power-of-2 multiply with shift

- Operation
  - `u << k` gives `u * ` ***2^k***
  - Both signed and unsigned

Operands: *w* bits

True Product: *w+k* bits

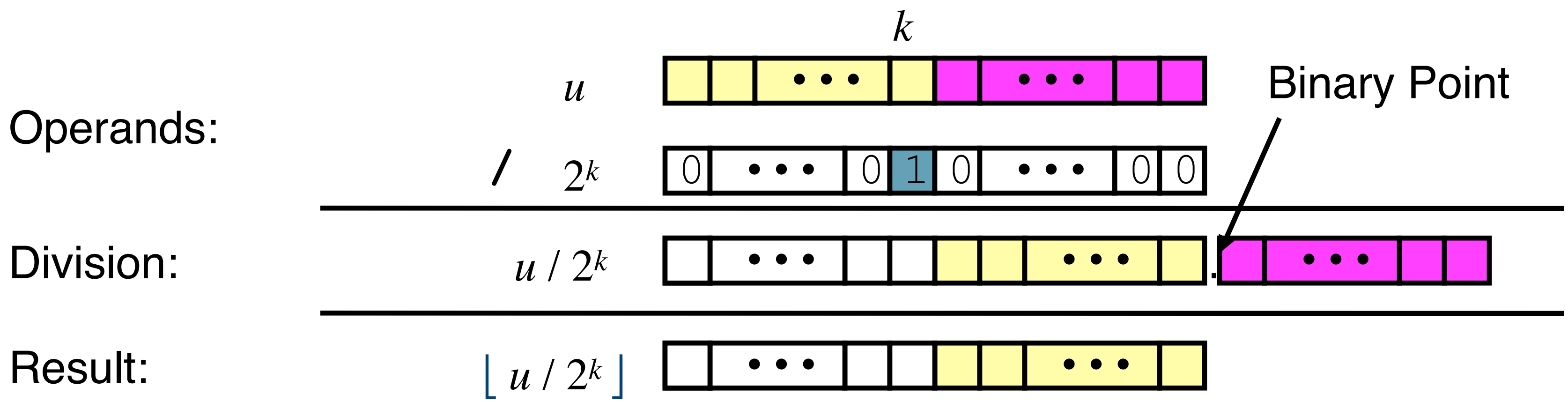
Discard *k* bits: *w* bits



- Examples
  - `3*a = a<<1 + a`
  - Most machines shift and add much faster than multiply (1 to +12 cycles)
    - Compiler generates this code automatically

Unsigned power-of-2 divide with shift

- Quotient of unsigned by power of 2
  - `u >> k` gives $\lfloor$ `u / ` $\boldsymbol{2^k}$ $\rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	**0**`0011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	**0000**`0011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	**00000000** `00111011`

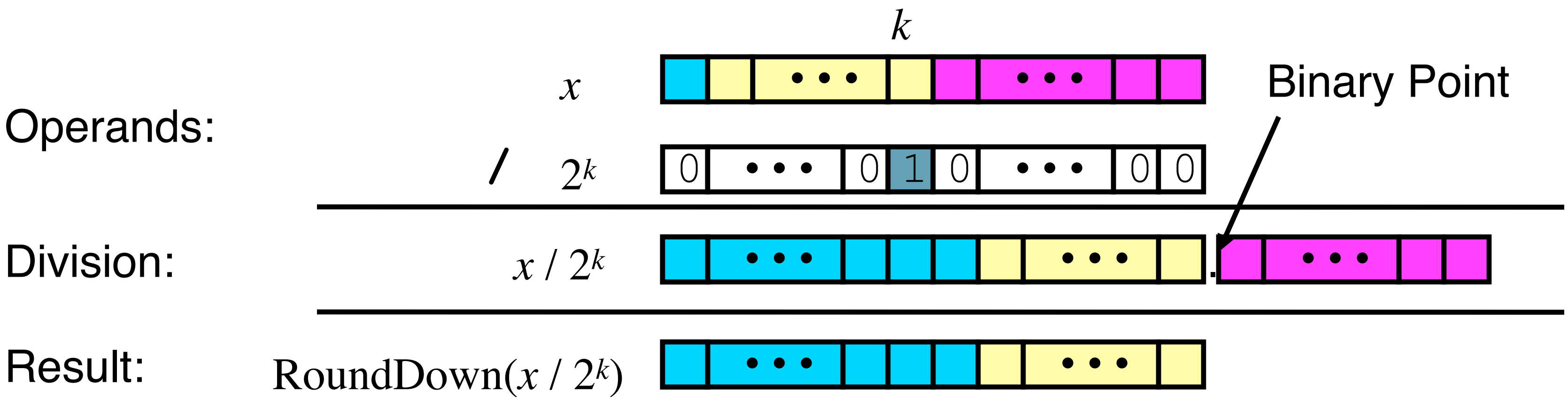
Arithmetic Right Shift = Division by 2?

- Compare right-shifting 3-bit negative numbers to dividing by 2

100	-4
101	-3
110	-2
111	-1
000	0
001	1
010	2
011	3

Signed power-of-2 divide with shift

- Quotient of signed by power of 2
  - `x >> k` gives $\lfloor$ `x /` 2^k $\rfloor$
  - Uses arithmetic shift
  - Rounds wrong direction when `u < 0`

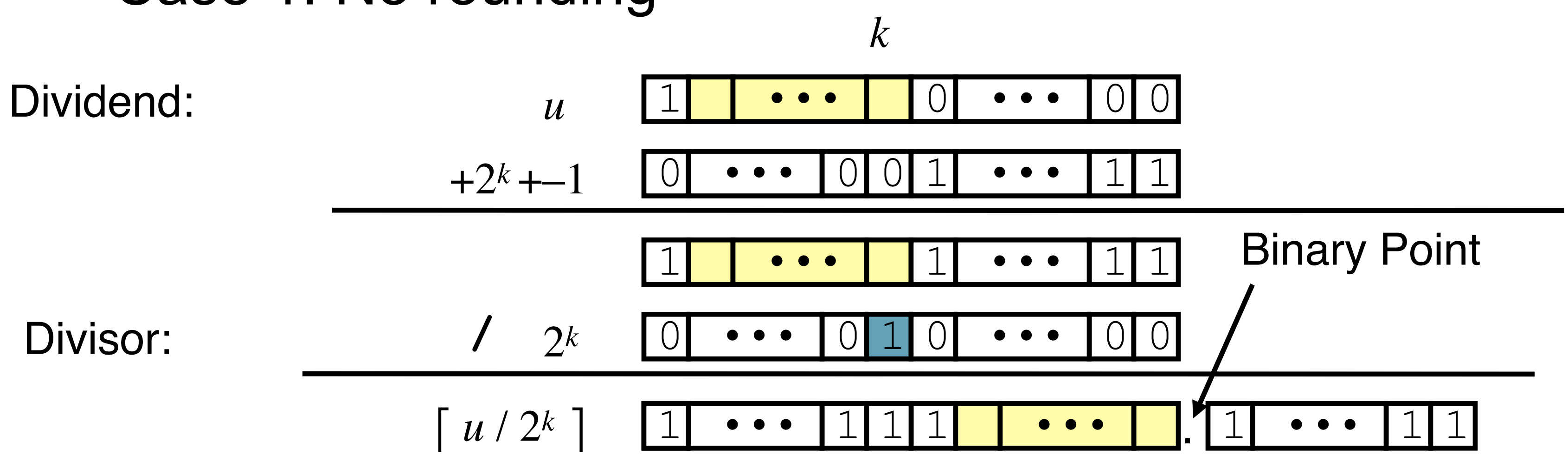


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct power-of-2 divide

- Quotient of negative number by power of 2
  - Want $\lceil \texttt{x} / 2^k \rceil$ (Round Toward 0)
  - Compute as $\lfloor (\texttt{x}+2^k-1) / 2^k \rfloor$
    - In C: `(x<0 ? (x + (1<<k) - 1) : x) >> k`
    - Biases dividend toward 0

- Case 1: No rounding



Biasing has no effect

Correct power-of-2 divide (Cont.)

Case 2: Rounding